<u>The Principle of Strong Mathematical Induction (2nd Principle)</u>

Let a and b be fixed positive integers such that $a \leq b$, and
let $P(n)$ be a predicate for all integers $n \geq a$.

IF

1) P(a), P(a+1), . . . , P(b) are all true, and
2) For every integer $k \geq b$,

   If P(m) is true for every integer m such that $a \leq m \leq k$,
   then P(k+1) is true,

THEN

P(n) is true for every integer n such that $n \geq a$.

Strong Mathematical Induction (in Diagram Form) with $a = 1$:

Given an infinite list of statements: $P_1, P_2, P_3, P_4,$ . . .

IF we prove that:

<u>These</u> are true, and that, for all integers $k \geq b$, (so $k+1 \geq b+1$)

if all the P_m, with $1 \leq m \leq k$, are true, | then P_{k+1} is true,

. . .   .

$P_1, P_2,$ . . . , $P_{\mathbf{b}}, P_{\mathbf{b+1}},$ . . . , $P_m,$ . . . , $P_{k-1}, P_k, P_{k+1},$ . . .

THEN we can logically conclude that <u>ALL of these</u> statements are true.

. . .   . . .

$P_1, P_2, P_3, P_4,$ . . . , $P_{k-1}, P_k, P_{k+1},$ . . .

The Design for Proofs using the Principle of Strong Mathematical Induction (2nd Principle)
(a represents a particular integer.)

To Prove: For every integer n such that $n \geq a$, predicate P(n) .

Proof: (by Strong Mathematical Induction)

[**Basis Step: Show that P(n) is true when n = a, a + 1, a + 2, …, b for an appropriate number b $\geq$ a.]**

. . . [See below for a format for the Basis Step.] **[End of Inductive Step]**

[Inductive Step:

To Prove: For every integer k $\geq$ b,

if P(m) is true for every integer m such that a $\leq$ m $\leq$ k , then P(k+1) is true.]

Let k be any integer such that $k \geq b$.

Suppose P(m) , for every integer m such that $a \leq m \leq k$. [This is the ***Inductive Hypothesis***.]

[NTS: P(k+1)]

. . . (proof statements come here,
including statements that apply the Inductive Hypothesis for various integers t as needed,
that is, statements similar to the following :

"Since $\boldsymbol{a} \leq \mathbf{t} \leq \mathbf{k}$, P(**t**) , by the Inductive Hypothesis."
[Note: $\boldsymbol{a} \leq \mathbf{t} \leq \mathbf{k}$ needs to be proved first!])

. . .

$\therefore$ P(k+1) .

$\therefore$ For every integer k such that $k \geq b$, if P(m) , for every integer m such that $a \leq m \leq k$, then P(k+1), by Direct Proof. **[End of Inductive Step]**

$\therefore$ P(n) for every integer n such that $n \geq a$, **by Strong Mathematical Induction**. **QED**

A Format for the Basis Step

Let n = a; . . . (calculations with n = a) . . . ; $\therefore$ For n = a, P(n) .
Let n = a + 1; . . . (calculations with n = a + 1) . . . ; $\therefore$ For n = a + 1, P(n) .
Let n = a + 2; . . . (calculations with n = a + 2) . . . ; $\therefore$ For n = a + 2, P(n) .
. . . .
Let n = b; . . . (calculations with n = b) . . . ; $\therefore$ For n = b, P(b) . [End of Basis Step]

Notes:

1) The number of initial cases of the predicate that need to be verified in the Basis Step depends on the nature of the predicate P(n) . Sometimes there will be only one initial case to verify. Usually, however, there are two or more cases. The number of cases that need to be verified in the Basis Step often equals the number of previous cases (P(k), P(k-1), P(k-2), etc.) that need to be accessed in order to prove P(k+1) .

2) The supposition " Suppose that P(m) is true for every integer m such that $a \leq m \leq k$ " is how mathematicians say
"Suppose that P(k) and all of the cases of the predicate before P(k) are true."

This supposition is called *the Inductive Hypothesis*. When justifying a conclusion by this supposition, you indicate this by saying: " $\therefore$ Such-and-such is true, ***by the Inductive Hypothesis***. "

(Pay attention to the different ways that the Inductive Hypothesis is worded in two principles of Mathematical Induction !)

3) Before you can apply the Inductive Hypothesis to conclude that P(t) is true, for a particular value of t , you must first prove that $a \le t \le k$.

Example Proofs by Strong Mathematical Induction

The first example proof is a proof of a conjecture dealing with a particular sequence of real numbers. First, a word about a method of defining a sequence of real numbers that may be new to you.

Some sequences $(d_n)_{n = 1, 2, 3, \dots}$ are defined recursively, that is, after a certain point, each term is defined by some formula involving previous terms of the sequence.

Strong Mathematical Induction is useful for proving conjectures about such recursively defined (also called "inductively defined") sequences .

Define the sequence $(d_n)_{n = 1, 2, 3, \dots}$ as follows:

Define the **first term** as $d_1 = \frac{9}{10}$. Define the **second term** $d_2 = \frac{10}{11}$.

For all integers $k \ge 3$, define d_k by the formula, $d_k = d_{(k-1)} \times d_{(k-2)}$.

Thus, $d_3 = d_2 \times d_1 = \frac{10}{11} \times \frac{9}{10} = \frac{9}{11}$ and $d_4 = d_3 \times d_2 = \frac{9}{11} \times \frac{10}{11} = \frac{90}{121}$. And so on, . . .

This process defines the complete sequence $d_1 , d_2 , d_3 , d_4 , \dots$

In a proof of a conjecture about this sequence (d_n) (and other such recursively defined sequences), before you can apply the defining formula to conclude that $d_t = d_{(t-1)} \times d_{(t-2)}$ for a particular value of t , you must first prove that $t \ge 3$ so that we know that the formula applies.

An Example proof of a conjecture about this sequence using Strong Mathematical Induction

[Recall, $d_1 = \frac{9}{10}$. $d_2 = \frac{10}{11}$. For all integers $k \ge 3$, $d_k = d_{(k-1)} \times d_{(k-2)}$.]

To Prove: **For every integer $n \ge 1$, d_n is such that $0 < d_n < 1$.**

Proof: [by Strong Mathematical Induction] [Here, P(n) is the predicate: " $0 < d_n < 1$ ".]

[Basis Step: To Prove: For n = 1 and n = 2 , $0 < d_n < 1$.] [Here, a = 1 and b = 2]

[Note: Since the definition of d_k , for $k \ge 3$, depends on terms in the sequence **two (2)** positions back, we will need to verify **two (2)** initial cases of the predicate in the Basis Step.
Also, since the formula definition for d_k requires $k \ge 3$, we will need to verify P(n) for

n = 1 and for n = 2 , so that when we set $k \geq 2$ in the Inductive Step, we will have that $(k+1) \geq 3$ and we will be able to apply the defining formula to $d_{(k+1)}$.]

Let n = 1 . $d_n = d_1 = \frac{9}{10}$ and $0 < \frac{9}{10} < 1$. ∴ For n = 1, $0 < d_n < 1$.

Let n = 2 . $d_n = d_2 = \frac{10}{11}$ and $0 < \frac{10}{11} < 1$. ∴ For n = 2, $0 < d_n < 1$.

[End of Basis Step]

[**Inductive Step:**

To Prove: For every integer $k \geq 2$,

if $0 < d_m < 1$ for every integer m such that $1 \leq m \leq k$, then $0 < d_{k+1} < 1$.]

Let k be any integer such that $k \geq 2$.

[Inductive Hypothesis: "Suppose P(m) , for every integer m such that $a \leq m \leq k$. "]

Suppose that $0 < d_m < 1$, for every integer m such that $1 \leq m \leq k$. [The I. H.]

[We need to prove P(k+1), that is, " $0 < d_{(k+1)} < 1$ "]

Since $k \geq 2$, $(k+1) \geq 3$ and so, $3 \leq (k+1)$.

Since $(k+1) \geq 3$, $d_{(k+1)} = (d_k \times d_{(k-1)})$, by the defining formula of the sequence.

Since $3 \leq (k+1)$, $(3-2) \leq (k+1)-2$. ∴ $1 \leq k-1 \leq k$.

Since $1 \leq k-1 \leq k$, $0 < d_{(k-1)} < 1$, by the Inductive Hypothesis .

Since $1 \leq k \leq k$, $0 < d_k < 1$, by the Inductive Hypothesis .

Since $0 < d_{(k-1)} < 1$, $(d_k \times 0) < (d_k \times d_{(k-1)}) < d_k \times 1 = d_k$.

∴ $0 < (d_k \times d_{(k-1)}) < d_k < 1$, and so, by substitution .

∴ $0 < (d_k \times d_{(k-1)}) < 1$, by transitivity of "$<$ " .

∴ $0 < d_{(k+1)} < 1$ by substitution, since $d_{(k+1)} = (d_k \times d_{(k-1)})$. [∴ P(k+1)]

∴ For every integer $k \geq 2$, if $0 < d_m < 1$, for every integer m such that $1 \leq m \leq k$, then $0 < d_{(k+1)} < 1$, by Direct Proof .

[End of Inductive Step]

∴ **$0 < d_n < 1$ for every integer $n \geq 1$, by Strong Mathematical Induction.**

Q E D

Another Example Proof by Strong Mathematical Induction

Theorem 4.3.4: (p. 138): Every positive integer greater than 1 has at least one prime divisor;

that is, for every integer $n > 1$, there exists a prime number p such that $p \mid n$.

[We prove equivalently that the predicate is true for every integer $n \geq 2$.]

Proof: [by Strong Mathematical Induction]

[Here, P(n) is the predicate:
"there exists a prime number p such that $p \mid n$," or equivalently,
"there exists a prime number q such that $q \mid n$."]

[We use Strong Mathematical Induction to prove "P(n) is true for every integer $n \geq 2$ " ,
which is equivalent to the statement "P(n) is true for every integer $n > 1$ ".]

[**Basis Step**: To Prove: For $n = 2$, there exists a prime number p such that $p \mid 2$.] [Here, $a = 2$ & $b = 2$.]

Let $n = 2$.

The integer 2 is a prime number and $n = 2 = 2\times1$, so $2 \mid n$.
∴ For $n = 2$, n is divisible by the prime number 2 .

∴ For $n = 2$, there exists a prime number p such that $p \mid n$. [Here, $p = 2$.]

[**End of Basis Step.** Here, since only one initial case needs to be verified, $b = a = 2$.]

[Inductive Step:

To Prove: For every integer $k \geq 2$,

if, for every integer m such that $2 \leq m \leq k$,
there exists a prime number q such that $q \mid m$,
then there exists a prime number p such that $p \mid (k+1)$.]

Let k be any integer such that $k \geq 2$.

Suppose that, for every integer m such that $2 \leq m \leq k$,
there exists a prime number q such that $q \mid m$. [Inductive Hypothesis]

[We need to prove P(k+1), that is, prove " k+1 is divisible by some prime number p ".]

Now, **k+1 is prime or k+1 is not prime.** [There are two possibilities.]

Case 1: ((k+1) is a prime number.)

Suppose that k+1 is a prime number.

The integer k+1 is a prime number and $k+1 = (k+1)(1)$, so $(k+1) \mid (k+1)$.

∴ There exists a prime number p such that $p \mid (k+1)$ in Case 1. [Here, $p = k+1$.]

[**End of Case 1**]

Case 2: ((k+1) is not a prime number.)

Suppose that k+1 is not a prime number.

Because the integer k+1 is not a prime number, there exist integers r and s such that

$$(k+1) = rs \quad \text{and} \quad 1 < r < (k+1) \quad \text{and} \quad 1 < s < (k+1).$$

$\therefore$ r | (k+1), since (k+1) = r s and s is an integer.

Since $1 < r < (k+1)$, we conclude that **2 ≤ r ≤ k**.

[Recall that we just proved that **2 ≤ r ≤ k**, which we needed in order to apply the Inductive hypothesis.]

By the Inductive Hypothesis, since 2 ≤ r ≤ k ,
there exists a prime number p such that p | r .

Thus, p | r and r | (k+1) .

p | (k+1) , by the transitivity of divisibility , and p is a prime number .

$\therefore$ There exists a prime number p such that p | (k+1) in Case 2.
[**End of Case 2**]

$\therefore$ **There exists a prime number p such that p | (k+1) , in general.** [$\therefore$ P(k+1)]

$\therefore$ For every integer k such that $k \geq 2$,
if ,
for every integer m such that 2 ≤ m ≤ k, there exists a prime q such that q | m ,
then
there exists a prime p such that p | (k+1) , by Direct Proof .

[End of Inductive Step]

$\therefore$ For every integer n > 1 (that is, for every integer $n \geq 2$),
there exists a prime number p such that p | n , by Strong Mathematical Induction.

Q E D